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SOLUTION OF THE PLANE MIXED PROBLEM OF THE THEORY OF ELASTICITY IN THE FORM OF A SERIES IN LEGENDRE POLYNOMIALS

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By using Legendre polynomials, it is possible to demonstrate a second procedure, different from those published in the literature [1, 2], for reducing the problem to the solution of an algebraic system or to the solution of a boundary-value problem for ordinary differential equations.

1. Formulation of the Problem. The plane mixed boundary-value problem of the theory of elasticity consists in finding the functions p, q, τ, u, and v satisfying the equations

$$\partial p/\partial x + \partial\tau/\partial y + \gamma_1 = 0,\ \partial\tau/\partial x + \partial q/\partial y + \gamma_2 = 0,$$
$$p - \alpha\partial u/\partial x - \beta\partial v/\partial y = 0,\ q - \alpha\partial v/\partial y - \beta\partial u/\partial x = 0,$$
$$\tau - \mu(\partial u/\partial y + \partial v/\partial x) = 0,\ \alpha = 2\mu\,(1-\nu)/(1-2\nu),\ \nu < 1/2,\quad \mu > 0.$$
$$\beta = \alpha\nu/(1-\nu)$$

within some region Ω and taking on specified values on the boundary of the region. We shall confine ourselves to the case in which Ω is a square, $\Omega = \{x, y \mid x\in[-1,1], y\in[-1,1]\}$, and the boundary conditions are such that by a transformation of the desired functions the problem can be reduced to finding the functions p, q, τ, u, and v satisfying the zero boundary conditions

$$(pu)_{x=\pm1} = (qv)_{y=\pm1} = (\tau v)_{x=\pm1} = (\tau u)_{y=\pm1} = 0 \tag{1.1}$$

and the equations

$$\partial p/\partial x + \partial\tau/\partial y + f_1 = 0,\ \partial\tau/\partial x + \partial q/\partial y + f_2 = 0,$$
$$p - \alpha\partial u/\partial x - \beta\partial v/\partial y + f_3 = 0,\ q - \alpha\partial v/\partial y - \beta\partial u/\partial x + f_4 = 0,$$
$$\tau - \mu\,(\partial u/\partial y + \partial v/\partial x) + f_5 = 0,$$

where the f_σ ($\sigma = 1, \ldots, 5$) are known functions which are quadratic summable over Ω. We assume that in each of the equations (1.1) one of the multiplied functions is equal to zero all along one side of the square.

If in the case of a displacement of the square as an absolutely rigid body

$$u = a + \omega y,\ v = b - \omega x$$

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(a, b, and ω are constants) from the boundary condition (1.1) it does not follow that

$$a = b = \omega = 0, \tag{1.2}$$

then we shall supplement the conditions (1.1) with those of the equations

$$\int_\Omega u d\Omega = 0, \ \int_\Omega v d\Omega = 0, \ \int_\Omega (uy - vx)\, d\Omega = 0, \tag{1.3}$$

which, together with the conditions (1.1), ensure that the equations (1.2) will be satisfied. In those cases in which we use any of the equations (1.3), the functions f_1 and f_2 cannot be arbitrary. We associate with the equations (1.3) the following equations:

$$\int_\Omega f_1 d\Omega = 0, \ \int_\Omega f_2 d\Omega = 0, \ \int_\Omega (f_1 y - f_2 x)\, d\Omega = 0. \tag{1.4}$$

When we use any of the equations (1.3), the functions f_1 and f_2 must satisfy the corresponding equations in (1.4).

2. Approximate Solution. We write

$$\begin{aligned}
p^{nm} &= \sum_{k=0}^{n} \sum_{i=0}^{m} p_{ki}^{nm} P_k Q_i, \quad q^{nm} = \sum_{k=0}^{n} \sum_{i=0}^{m} q_{ki}^{nm} P_k Q_i,\\
\tau_1^{nm} &= \sum_{k=0}^{n-1} \sum_{i=0}^{m+1} \tau_{ki}^{nm} P_k Q_i, \quad \tau_2^{nm} = \sum_{k=0}^{n+1} \sum_{i=0}^{m-1} \tau_{ki}^{nm} P_k Q_i,\\
u_0^{nm} &= \sum_{k=0}^{n} \sum_{i=0}^{m-1} u_{ki}^{nm} P_k Q_i, \quad v_0^{nm} = \sum_{k=0}^{n-1} \sum_{i=0}^{m} v_{ki}^{nm} P_k Q_i,\\
u_1^{nm} &= \sum_{k=0}^{n} \sum_{i=0}^{m+1} u_{ki}^{nm} P_k Q_i, \quad v_1^{nm} = \sum_{k=0}^{n+1} \sum_{i=0}^{m} v_{ki}^{nm} P_k Q_i,\\
u_2^{nm} &= \sum_{k=0}^{n+2} \sum_{i=0}^{m-1} u_{ki}^{nm} P_k Q_i, \quad v_2^{nm} = \sum_{k=0}^{n-1} \sum_{i=0}^{m+2} v_{ki}^{nm} P_k Q_i,
\end{aligned} \tag{2.1}$$

where $n, m \geqslant 1$; $p_{ki}^{nm}, q_{ki}^{nm}, \tau_{ki}^{nm}, u_{ki}^{nm}, v_{ki}^{nm}$ are constants; $P_k = P_k(y)$, $Q_i = Q_i(x)$ are Legendre polynomials [3] orthogonal in the interval $[-1, 1]$; and k, i are the degrees of the polynomials.

We require the functions (2.1) to satisfy the zero boundary conditions

$$(p^{nm} u_1^{nm})_{x=\pm 1} = (q^{nm} v_1^{nm})_{y=\pm 1} = (\tau_1^{nm} v_2^{nm})_{x=\pm 1} = (\tau_2^{nm} u_2^{nm})_{y=\pm 1} = 0 \tag{2.2}$$

and the equations

$$\begin{gathered}
\int_\Omega \left(\frac{\partial p^{nm}}{\partial x} + \frac{\partial \tau_2^{nm}}{\partial y} + f_1 \right) P_k Q_i d\Omega = 0, \ k = 0, 1, \ldots, n, \ i = 0, 1, \ldots, m-1;\\
\int_\Omega \left(\frac{\partial \tau_1^{nm}}{\partial x} + \frac{\partial q^{nm}}{\partial y} + f_2 \right) P_k Q_i d\Omega = 0, \ k = 0, 1, \ldots, n-1, \ i = 0, 1, \ldots, m;\\
\int_\Omega \left(p^{nm} - \alpha \frac{\partial u_1^{nm}}{\partial x} - \beta \frac{\partial v_1^{nm}}{\partial y} + f_3 \right) P_k Q_i d\Omega = 0,\\
\int_\Omega \left(q^{nm} - \alpha \frac{\partial v_1^{nm}}{\partial y} - \beta \frac{\partial u_1^{nm}}{\partial x} + f_4 \right) P_k Q_i d\Omega = 0,\\
k = 0, 1, \ldots, n, \quad i = 0, 1, \ldots, m;\\
\int_\Omega \left[\tau_2^{nm} - \mu \left(\frac{\partial u_2^{nm}}{\partial y} + \frac{\partial v_2^{nm}}{\partial x} \right) + f_5 \right] P_k Q_i d\Omega = 0, \ k = 0, 1, 2, \ldots, n+1, \ i = 0, 1, 2, \ldots, m-1;\\
\int_\Omega \left[\tau_1^{nm} - \mu \left(\frac{\partial u_2^{nm}}{\partial y} + \frac{\partial v_2^{nm}}{\partial x} \right) + f_5 \right] P_k Q_i d\Omega = 0, k = 0, 1, 2, \ldots, n-1, \quad i = 0, 1, 2, \ldots, m+1.
\end{gathered} \tag{2.3}$$

We assume that in each of the equations (2.2) one of the factors [the same one as in (1.1)] is zero over an entire side of the square.

If the formulation of the problem contains any of the equations (1.3), then the system (2.2), (2.3) will be supplemented by the corresponding equations

$$u_{00}^{nm}=0,\ v_{00}^{nm}=0,\ u_{10}^{nm}-v_{01}^{nm}=0 \tag{2.4}$$

and we will eliminate from (2.3) those of the equations

$$\int_\Omega\left(\frac{\partial p^{nm}}{\partial x}+\frac{\partial\tau_2^{nm}}{\partial y}+f_1\right)P_j d\Omega=0,\ j=0,1,\ \int_\Omega\left(\frac{\partial\tau_1^{nm}}{\partial x}+\frac{\partial q^{nm}}{\partial y}+f_2\right)d\Omega=0,$$

which are the consequence of the remaining equations of the system (2.2), (2.3), and the equations (1.4).

The equations (2.2), (2.3), together with the corresponding equations (2.4), form a closed system for the functions (2.1). The solution of this system will be called the approximate solution. The tangential stress in the approximate solution can be calculated by the formula

$$\tau^{nm}=\sum_{k=0}^{n+1}\sum_{i=0}^{m-1}\tau_{ki}^{nm}P_kQ_i+\sum_{k=0}^{n-1}\sum_{i=m}^{m+1}\tau_{ki}^{nm}P_kQ_i.$$

The function τ^{nm} satisfies the equations (2.3) if instead of τ_1^{nm}, τ_2^{nm} we write τ^{nm}. The boundary conditions are approximately satisfied by τ^{nm}.

3. Energy Property of the Approximate Solution. We assume that the approximate solution exists. Making use of (2.2), (2.3), and the obvious equations of the type

$$\int_\Omega\frac{\partial p^{nm}}{\partial x}u_0^{nm}d\Omega=\int_\Omega\frac{\partial p^{nm}}{\partial x}u_1^{nm}d\Omega,$$

we can find

$$\int_\Omega\left[f_1u_0^{nm}+f_2v_0^{nm}+f_3\frac{\partial u_1^{nm}}{\partial x}+f_4\frac{\partial v_1^{nm}}{\partial y}+f_5\left(\frac{\partial u_2^{nm}}{\partial y}+\frac{\partial v_2^{nm}}{\partial x}\right)\right]d\Omega=E_{nm}, \tag{3.1}$$

where

$$E_{nm}=G_1(u_1^{nm},v_1^{nm})+G_2(u_2^{nm},v_2^{nm});$$

$$G_1(\varphi,\psi)=\int_\Omega\left[\alpha\left(\frac{\partial\varphi}{\partial x}\right)^2+2\beta\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y}+\alpha\left(\frac{\partial\psi}{\partial y}\right)^2\right]d\Omega;$$

$$G_2(\varphi,\psi)=\int_\Omega\mu\left(\frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial x}\right)^2d\Omega.$$

4. Estimate of Displacements Caused by Deformation. Let u be a function belonging to $L_2(\Omega)$ and having a generalized derivative $\partial u/\partial x$ belonging to $L_2(\Omega)$ [4]. We denote by $u_{k,i}$, $a_{k,i}$ the Fourier coefficients of the functions u, $\partial u/\partial x$

$$u\sim\sum_{k,i=0}^{\infty}u_{k,i}P_kQ_i,\ \frac{\partial u}{\partial x}\sim\sum_{k,i=0}^{\infty}a_{k,i}P_kQ_i.$$

Making use of the property of Legendre polynomials [3] that

$$\frac{d}{dx}(Q_{s+1}-Q_{s-1})=(1+2s)Q_s,\ s=1,2,\dots, \tag{4.1}$$

we find that

$$\int_{\Omega} \frac{\partial u}{\partial x} P_r \, (Q_{s+1} - Q_{s-1})\, d\Omega = (1+2s) \int_{\Omega} u P_r Q_s d\Omega,$$

and, consequently,

$$u_{r,s} = [1/(2s-1)] a_{r,s-1} - [1/(2s+3)] a_{r,s+1}, \quad r = 0, 1, 2, \ldots, \; s = 1, 2, \ldots .$$

Obviously,

$$\sum_{s=1}^{\infty} \frac{1}{1+2s} u_{r,s}^2 \leqslant a_{r,0}^2 + \frac{1}{2} \sum_{s=2}^{\infty} \frac{1}{2s-1} a_{r,s-1}^2 + \frac{1}{2} \sum_{s=1}^{\infty} \frac{1}{2s+3} a_{r,s+1}^2,$$

and therefore

$$\left\| u - \sum_{k=0}^{\infty} u_{k,0} P_k \right\| \leqslant \left\| \frac{\partial u}{\partial x} \right\|. \tag{4.2}$$

In (4.2) and below, the symbol $\| \; \|$ denotes the norm in $L_2(\Omega)$. Making use of (4.2) and the positive-definiteness of the integrand in the functional G_1, we find

$$\left\| u_1^{nm} - \sum_{k=0}^{n} u_{k0}^{nm} P_k \right\|^2 + \left\| v_1^{nm} - \sum_{i=0}^{m} v_{0i}^{nm} Q_i \right\|^2 \leqslant CG\left(u_1^{nm}, v_1^{nm}\right). \tag{4.3}$$

In (4.3) and below, the letter C denotes a constant independent of n and m.

According to (4.1),

$$\int_{\Omega} \left(\frac{\partial u_2^{nm}}{\partial y} + \frac{\partial v_2^{nm}}{\partial x} \right) (P_{r+1} - P_{r-1})(Q_{s+1} - Q_{s-1})\, d\Omega =$$

$$= \int_{\Omega} \left[(1+2r)\, u_2^{nm} P_r (Q_{s+1} - Q_{s-1}) + (1+2s)\, v_2^{nm} Q_s (P_{r+1} - P_{r-1})\right] d\Omega, \tag{4.4}$$

$$s, r = 1, 2, \ldots .$$

We write

$$\frac{\partial u_2^{nm}}{\partial y} + \frac{\partial v_2^{nm}}{\partial x} = \sum_{k=0}^{n+1} \sum_{i=0}^{m+1} b_{k,i}^{nm} P_k Q_i .$$

In (4.4) we set s = 1, obtaining

$$u_{r0}^{nm} = \frac{1}{2r-1} b_{r-1,0}^{nm} - \frac{1}{2r+3} b_{r+1,0}^{nm} - \frac{1}{5(2r-1)} b_{r-1,2}^{nm} +$$

$$+ \frac{1}{5(2r+3)} b_{r+1,2}^{nm} + \frac{1}{5} u_{r2}^{nm} - \frac{1}{2r-1} v_{r-11}^{nm} + \frac{1}{2r+3} v_{r+11}^{nm}, \; r = 1, 2, \ldots,$$

and, consequently,

$$\left(u_{10}^{nm} + v_{01}^{nm}\right)^2 \leqslant \frac{1}{5}\left[\left(u_{12}^{nm} + v_{21}^{nm}\right)^2 + 25\left(b_{0,0}^{nm}\right)^2 + \left(b_{2,0}^{nm}\right)^2 + \left(b_{0,2}^{nm}\right)^2 + \frac{1}{9}\left(b_{2,2}^{nm}\right)^2\right],$$

$$\sum_{r=2}^{n} \frac{1}{2r+1} \left(u_{r0}^{nm}\right)^2 \leqslant \sum_{r=0}^{n+1} \frac{1}{2r+1} \left[\left(b_{r,0}^{nm}\right)^2 + \frac{1}{5}\left(b_{r,2}^{nm}\right)^2\right] + \frac{7}{5}\left[\frac{1}{5} \sum_{r=2}^{n} \frac{1}{2r+1} \left(u_{r2}^{nm}\right)^2 + \frac{1}{3} \sum_{r=1}^{n} \frac{1}{2r+1} \left(v_{r1}^{nm}\right)^2\right]. \tag{4.5}$$

From (4.3) and (4.5) we find

$$\left(u_{10}^{nm} + v_{01}^{nm}\right)^2 \leqslant CE_{nm}, \quad \left\| \sum_{k=2}^{n} u_{k0}^{nm} P_k \right\|^2 \leqslant CE_{nm}. \tag{4.6}$$

In an analogous manner, we obtain the estimate

$$\left\| \sum_{i=2}^{m} v_{0i}^{nm} Q_i \right\|^2 \leqslant CE_{nm}. \tag{4.7}$$

From (4.3), (4.6), and (4.7) it follows that

$$\left\| u_1^{nm} - \sum_{k=0}^{1} u_{k0}^{nm} P_k \right\|^2 + \left\| v_1^{nm} - \sum_{i=0}^{1} v_{0i}^{nm} Q_i \right\|^2 \leqslant CE_{nm},$$
$$(u_{10}^{nm} + v_{01}^{nm})^2 \leqslant CE_{nm}. \tag{4.8}$$

Obviously,

$$\| u_2^{nm} \|^2 = \| u_0^{nm} \|^2 + \left\| \sum_{i=0}^{m-1} u_{n+1i}^{nm} P_{n+1} Q_i \right\|^2 + \left\| \sum_{i=0}^{m-1} u_{n+2i}^{nm} P_{n+2} Q_i \right\|^2,$$
$$\left\| \frac{\partial u_2^{nm}}{\partial y} + \frac{\partial v_2^{nm}}{\partial x} \right\|^2 \geqslant \left\| \sum_{i=0}^{m-1} u_{n+1i}^{nm} (2n+1) P_n Q_i \right\|^2 + \left\| \sum_{i=0}^{m-1} u_{n+2i}^{nm} (2n+3) P_{n+1} Q_i \right\|^2. \tag{4.9}$$

From (4.8) and (4.9) it follows that

$$\left\| u_2^{nm} - \sum_{k=0}^{1} u_{k0}^{nm} P_k \right\|^2 \leqslant CE_{nm}. \tag{4.10}$$

In an analogous manner, we obtain the estimate

$$\left\| v_2^{nm} - \sum_{i=0}^{1} v_{0i}^{nm} Q_i \right\|^2 \leqslant CE_{nm}. \tag{4.11}$$

The inequalities (4.8), (4.10), and (4.11) give us an estimate in the approximate solution for the deformation-caused displacements in terms of the energy of elastic deformation.

From the proof of the inequalities (4.8) we can see that for any functions u, $v \in L_2(\Omega)$, which have generalized derivatives $\partial u/\partial x$, $\partial v/\partial y \in L_2(\Omega)$ and a generalized sum of derivatives $(\partial u/\partial y + \partial v/\partial x) \in L_2(\Omega)$, the following inequalities hold:

$$\left\| u - \sum_{k=0}^{1} u_{k0} P_k \right\|^2 + \left\| v - \sum_{i=0}^{1} v_{0i} Q_i \right\|^2 \leqslant CE(u, v),$$
$$(u_{10} + v_{01})^2 \leqslant CE(u, v), \tag{4.12}$$

where

$$u_{k0} = \frac{1}{4}(1 + 2k) \int_{\Omega} u P_k d\Omega; \; v_{0i} = \frac{1}{4}(1 + 2i) \int_{\Omega} v Q_i d\Omega;$$
$$E(u, v) = G_1(u, v) + G_2(u, v).$$

By the generalized sum of the derivatives $\partial u/\partial y + \partial v/\partial x$ of the functions u, v we mean a function $\psi \in L_2(\Omega)$, for which the inequality

$$\int_{\Omega} \left(\psi\varphi + u \frac{\partial \varphi}{\partial y} + v \frac{\partial \varphi}{\partial x} \right) d\Omega = 0$$

is satisfied, where φ is any function belonging to $W_2{}^1(\Omega)$ which is equal to zero along the sides of the square [4].

5. Estimate for "Rigid" Displacement. Let τ_*, u_*, and v_* be functions which belong to $W_2{}^1(\Omega)$ and satisfy the conditions

$$(p_* u_*)_{x=\pm 1} = (q_* v_*)_{y=\pm 1} = (\tau_* v_*)_{x=\pm 1} = (\tau_* u_*)_{y=\pm 1} = 0; \tag{5.1}$$

let p_* and q_* be functions which belong to $L_2(\Omega)$, have generalized derivatives $\partial p_*/\partial x$, $\partial q_*/\partial y \in L_2(\Omega)$, and satisfy the conditions (5.1). We assume that in each of the equations (5.1) one of the factors [the same one as in (1.1)] vanishes along the entire side of the square.

From (2.2) and (5.1) it follows that

$$\int_\Omega \left(\frac{\partial p_*}{\partial x} u_1^{nm} + p_* \frac{\partial u_1^{nm}}{\partial x} \right) d\Omega = 0, \quad \int_\Omega \left(\frac{\partial q_*}{\partial y} v_1^{nm} + q_* \frac{\partial v_1^{nm}}{\partial y} \right) d\Omega = 0,$$
$$\int_\Omega \left[\frac{\partial \tau_*}{\partial y} u_2^{nm} + \frac{\partial \tau_*}{\partial x} v_2^{nm} + \tau_* \left(\frac{\partial u_2^{nm}}{\partial y} + \frac{\partial v_2^{nm}}{\partial x} \right) \right] d\Omega = 0. \tag{5.2}$$

Distinguishing in (5.2) the terms corresponding to the displacement of the square as an absolutely rigid body, we can write (5.2) in the form

$$u_{00}^{nm} \int_{-1}^{1} (p_*)_{x=-1}^{x=1} dy + u_{10}^{nm} \int_{-1}^{1} (p_*)_{x=-1}^{x=1} y\,dy = F_1,$$
$$v_{00}^{nm} \int_{-1}^{1} (q_*)_{y=-1}^{y=1} dx + v_{01}^{nm} \int_{-1}^{1} (q_*)_{y=-1}^{y=1} x\,dx = F_2,$$
$$u_{00}^{nm} \int_{-1}^{1} (\tau_*)_{y=-1}^{y=1} dx + u_{10}^{nm} \int_{-1}^{1} (\tau_* y)_{y=-1}^{y=1} dx + v_{00}^{nm} \int_{-1}^{1} (\tau_*)_{x=-1}^{x=1} dy +$$
$$+ v_{01}^{nm} \int_{-1}^{1} (\tau_* x)_{x=-1}^{x=1} dy = F_3. \tag{5.3}$$

In (5.3) the F_i ($i = 1, 2, 3$) depend on p_*, q_*, τ_*, the derivatives of these functions, and the displacements caused by the deformation. Therefore, the F_i can be estimated in terms of the energy of elastic deformation, the norms of the functions p_*, q_*, τ_*, and the norms of their derivatives.

Selecting the functions p_*, q_*, τ_* in an appropriate manner and making use of the second inequality in (4.8) and those of the equations (2.4) that we use for completing the system (2.2), (2.3), we can prove that the following inequality holds:

$$\max \{|u_{00}^{nm}|, |u_{10}^{nm}|, |v_{00}^{nm}|, |v_{01}^{nm}|\} \leqslant C E_{nm}^{1/2}. \tag{5.4}$$

In an analogous manner, making use of the inequalities (4.12), we find that

$$\max \{|u_{00}|, |u_{10}|, |v_{00}|, |v_{01}|\} \leqslant C E(u, v) \tag{5.5}$$

for any functions u, $v \in L_2(\Omega)$, which have generalized derivatives $\partial u/\partial x$, $\partial v/\partial y \in L_2(\Omega)$, and a generalized sum of derivatives $(\partial u/\partial y + \partial v/\partial x) \in L_2(\Omega)$, and which satisfy the equations

$$\int_\Omega \left(\frac{\partial p_*}{\partial x} u + p_* \frac{\partial u}{\partial x} \right) d\Omega = 0, \quad \int_\Omega \left(\frac{\partial q_*}{\partial y} v + q_* \frac{\partial v}{\partial y} \right) d\Omega = 0,$$
$$\int_\Omega \left[u \frac{\partial \tau_*}{\partial y} + v \frac{\partial \tau_*}{\partial x} + \tau_* \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] d\Omega = 0 \tag{5.6}$$

and the equations in (1.3) that are used to supplement the conditions (1.1). In (5.5), u_{k0}, v_{k0} ($k = 0, 1$) have the same meaning as in (4.12).

6. Existence of an Approximate Solution. From (3.1), (4.8), (4.10), (4.11), (5.4), and (2.3) we find that the zero solution of the homogeneous system of equations of the approximate solution is unique and that, consequently, the determinant of this system is nonzero.

7. Generalized Solution. From (3.1), (4.8), (4.10), (4.11), (5.4), and (2.3) it follows that the norms in $L_2(\Omega)$ of the functions (2.1), of the derivatives

$$\frac{\partial u_1^{nm}}{\partial x}, \quad \frac{\partial v_1^{nm}}{\partial y} \tag{7.1}$$

and of the sums of the derivatives

$$\frac{\partial u_2^{nm}}{\partial y}+\frac{\partial v_2^{nm}}{\partial x},\quad \frac{\partial p^{nm}}{\partial x}+\frac{\partial \tau_2^{nm}}{\partial y},\quad \frac{\partial \tau_1^{nm}}{\partial x}+\frac{\partial q^{nm}}{\partial y} \tag{7.2}$$

are bounded uniformly with respect to n, m. Therefore, from any sequence of solutions (2.1) we can extract a sequence of solutions with numbers r, s that satisfies the following conditions:

1) the sequence converges as r, s → ∞ weakly in $L_2(\Omega)$ [4],

$$u_0^{rs},\ u_1^{rs},\ u_2^{rs}\to u;\quad \tau_1^{rs},\ \tau_2^{rs}\to\tau;$$
$$v_0^{rs},\ v_1^{rs},\ v_2^{rs}\to v;\quad p^{rs}\to p;\quad q^{rs}\to q;$$

2) the sequence of derivatives (7.1) converges weakly in $L_2(\Omega)$ to the generalized derivatives [4]

$$\frac{\partial u_1^{rs}}{\partial x}\to\frac{\partial u}{\partial x};\quad \frac{\partial v_1^{rs}}{\partial y}\to\frac{\partial v}{\partial y};$$

3) the sequence of sums of derivatives (7.2) converges to the generalized sums of derivatives

$$\frac{\partial u_2^{rs}}{\partial y}+\frac{\partial v_2^{rs}}{\partial x}\to\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x};\quad \frac{\partial p^{rs}}{\partial x}+\frac{\partial \tau_2^{rs}}{\partial y}\to\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial y};$$
$$\frac{\partial \tau_1^{rs}}{\partial x}+\frac{\partial q^{rs}}{\partial y}\to\frac{\partial \tau}{\partial x}+\frac{\partial q}{\partial y}.$$

From (2.3), by a passage to the limit as r, s → ∞, we find

$$\begin{gathered}\int_\Omega\left(\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial y}+f_1\right)\omega_1 d\Omega=0,\quad \int_\Omega\left(\frac{\partial \tau}{\partial x}+\frac{\partial q}{\partial y}+f_2\right)\omega_2 d\Omega=0,\\ \int_\Omega\left(p-\alpha\frac{\partial u}{\partial x}-\beta\frac{\partial v}{\partial y}+f_3\right)\omega_3 d\Omega=0,\\ \int_\Omega\left(q-\alpha\frac{\partial v}{\partial y}-\beta\frac{\partial u}{\partial x}+f_4\right)\omega_4 d\Omega=0,\\ \int_\Omega\left[\tau-\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+f_5\right]\omega_5 d\Omega=0,\end{gathered} \tag{7.3}$$

where ω_k (k = 1, ..., 5) are the derivatives of functions belonging to $L_2(\Omega)$.

From (2.2), (2.3), and (5.1) it follows that

$$\begin{gathered}\int_\Omega\left(p^{rs}\frac{\partial u_*}{\partial x}+\tau_2^{rs}\frac{\partial u_*}{\partial y}-f_1^{r,s-1}u_*\right)d\Omega=0,\\ \int_\Omega\left(\tau_1^{rs}\frac{\partial v_*}{\partial x}+q^{rs}\frac{\partial v_*}{\partial y}-f_2^{r-1,s}v_*\right)d\Omega=0,\\ f_\sigma^{r,s}=\frac{1}{4}\sum_{k=0}^{r}\sum_{i=0}^{s}(1+2k)(1+2i)\left(\int_\Omega f_\sigma P_k Q_i d\Omega\right)P_k Q_i.\end{gathered} \tag{7.4}$$

In (5.2) and (7.4) we pass to the limit as r, s → ∞ and find that the functions p, q, τ, u, v satisfy the equations (5.6) and the equations

$$\begin{gathered}\int_\Omega\left(p\frac{\partial u_*}{\partial x}+\tau\frac{\partial u_*}{\partial y}-f_1 u_*\right)d\Omega=0,\\ \int_\Omega\left(\tau\frac{\partial v_*}{\partial x}+q\frac{\partial v_*}{\partial y}-f_2 v_*\right)d\Omega=0.\end{gathered} \tag{7.5}$$

Obviously,

$$\lim_{r,s\to\infty}\left[G_1(u-u_1^{rs},\ v-v_1^{rs})+G_2(u-u_2^{rs},\ v-v_2^{rs})\right]=\lim_{r,s\to\infty}E_{rs}-E(u,v)\geqslant 0. \tag{7.6}$$

From (3.1) and (7.6) it follows that

$$\Phi(u,v)\geqslant E(u,v), \tag{7.7}$$

where

$$\Phi(\varphi,\psi)=\int_\Omega\left[f_1\varphi+f_2\psi+f_3\frac{\partial\varphi}{\partial x}+f_4\frac{\partial\psi}{\partial y}+f_5\left(\frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial x}\right)\right]d\Omega.$$

The functions p, q, τ, u, v satisfying Eqs. (5.6), (7.3), (7.5), and the inequality (7.7) form the generalized solution of the plane mixed problem of the theory of elasticity.

If in (7.3), we set

$$\omega_3=\frac{\partial u_*}{\partial x},\quad \omega_4=\frac{\partial v_*}{\partial y},\quad \omega_5=\frac{\partial u_*}{\partial y}+\frac{\partial v_*}{\partial x}$$

and make use of (7.5), we find that

$$2\Phi(u_*,\ v_*)=E(u_*\ v_*)-E(u-u_*,v-v_*)+E(u,v). \tag{7.8}$$

Assume that among the functions p_*, q_*, τ_*, u_*, v_* there are some which satisfy the equations (7.3) and those equations in (1.3) which are supplementary to the conditions (1.1). Substituting these functions into (7.3), setting

$$\omega_1=u,\ \omega_2=v,\ \omega_3=\partial u/\partial x,\ \omega_4=\partial v/\partial y,\ \omega_5=\partial u/\partial y+\partial v/\partial x$$

and making use of (5.6) and (7.7), we find that

$$2\Phi(u,v)=E(u,v)-E(u-u_*,v-v_*)+E(u_*,v_*)\geqslant 2E(u,v). \tag{7.9}$$

From (7.8) and (7.9) it follows that

$$E(u-u_*,\ v-v_*)=0. \tag{7.10}$$

According to (4.12) and (5.5), Eq. (7.10) has only a zero solution.

Thus, if the plane mixed problem of the theory of elasticity has a sufficiently smooth solution, then it coincides [in the sense of $L_2(\Omega)$] with the generalized solution, and it does so uniquely. From the uniqueness of the solution it follows that the entire sequence of solutions (2.1) converges to it weakly as m, n → ∞.

8. Reduction of the Problem to a Sequence of Boundary-Value Problems for Ordinary Differential Equations. We write

$$\begin{gathered}p_n=\sum_{k=0}^{n}p_k^nP_k,\quad q_n=\sum_{k=0}^{n}q_k^nP_k,\quad \tau_n'=\sum_{k=0}^{n-1}\tau_k^nP_k,\\ \tau_n''=\sum_{k=0}^{n+1}\tau_k^nP_k,\quad u_n'=\sum_{k=0}^{n}u_k^nP_k,\quad u_n''=\sum_{k=0}^{n+2}u_k^nP_k,\\ v_n'=\sum_{k=0}^{n+1}v_k^nP_k,\quad v_n''=\sum_{k=0}^{n-1}v_k^nP_k,\end{gathered} \tag{8.1}$$

where p_k^n, q_k^n, τ_k^n, u_k^n, v_k^n are functions of x; the $P_k = P_k(y)$ are Legendre polynomials; and k is the degree of the polynomial.

The approximate solution of the plane mixed problem of the thoery of elasticity will be sought in the form of functions (8.1) satisfying the boundary conditions

$$(q_n v_n')_{y=\pm 1} = (\tau_n'' u_n'')_{y=\pm 1} = 0; \tag{8.2}$$

$$(p_n^n u_n^n)_{x=\pm 1} = (p_k^n u_k^n)_{x=\pm 1} = (\tau_k^n v_k^n)_{x=\pm 1} = 0, \quad k = 0,1, \ldots, n-1, \tag{8.3}$$

and the equations

$$\begin{gathered}
\int_{-1}^{1} \left(\frac{\partial p_n}{\partial x} + \frac{\partial \tau_n''}{\partial y} + f_1^n \right) P_k dy = 0, \quad \int_{-1}^{1} \left(\frac{\partial \tau_n'}{\partial x} + \frac{\partial q_n}{\partial y} + f_2^{n-1} \right) P_k dy = 0, \\
\int_{-1}^{1} \left(p_n - \alpha \frac{\partial u_n'}{\partial x} - \beta \frac{\partial v_n'}{\partial y} + f_3^n \right) P_k dy = 0, \\
\int_{-1}^{1} \left(q_n - \alpha \frac{\partial v_n'}{\partial y} - \beta \frac{\partial u_n'}{\partial x} + f_4^n \right) P_k dy = 0, \\
\int_{-1}^{1} \left[\tau_n'' - \mu \left(\frac{\partial u_n''}{\partial y} + \frac{\partial v_n'}{\partial x} \right) + f_5^{n+1} \right] P_k dy = 0, \\
k = 0, 1, 2, \ldots, \\
\int_{-1}^{1} \left[\tau_n' - \mu \left(\frac{\partial u_n''}{\partial y} + \frac{\partial v_n''}{\partial x} \right) + f_5 \right] P_i dy = 0, \\
i = 0, 1, \ldots, n-1,
\end{gathered} \tag{8.4}$$

where f_σ^r is a segment of the series

$$f_\sigma^r = \sum_{k=0}^{n} f_{\sigma k} P_k, \quad f_{\sigma k} = \frac{1}{2}(1 + 2k) \int_{-1}^{1} f_\sigma P_k dy.$$

We assume that in each of the equations (8.2) one of the factors [the same one as in (1.1)] vanishes all along a side of the square.

If the formulation of the problem contains any of the equations (1.3), then the system (8.2), (8.4) is supplemented with the corresponding equations

$$\int_{-1}^{1} u_0^n dx = 0, \quad \int_{-1}^{1} v_0^n dx = 0, \quad \int_{-1}^{1} (u_1^n - 3v_0^n x)\, dx = 0. \tag{8.5}$$

Since the functions $\tau_1{}^{nm}$, $\tau_2{}^{nm}$, q^{nm}, $u_2{}^{nm}$ and the derivatives (7.1), (7.2) have norms which are uniformly bounded with respect to m, n, it follows that for fixed n the norms of the derivatives

$$\frac{\partial \tau_2^{nm}}{\partial y}, \frac{\partial p^{nm}}{\partial x}, \frac{\partial q^{nm}}{\partial y}, \frac{\partial \tau_1^{nm}}{\partial x}, \frac{\partial u_2^{nm}}{\partial y}, \frac{\partial v_2^{nm}}{\partial x}$$

will be bounded uniformly with respect to m. Therefore, from the solutions (2.1) we can form the subsequences

$$\begin{gathered}
u_1^{ns}, u_0^{ns} \to u_n'; \; u_2^{ns} \to u_n''; \; p^{ns} \to p_n; \; v_0^{ns}, v_2^{ns} \to v_n''; \; v_1^{ns} \to v_n'; \\
q^{ns} \to q_n; \; \tau_1^{ns} \to \tau_n'; \; \tau_2^{ns} \to \tau_n''; \; \frac{\partial p^{ns}}{\partial x} \to \frac{\partial p_n}{\partial x}; \; \frac{\partial q^{ns}}{\partial y} \to \frac{\partial q_n}{\partial y}; \\
\frac{\partial \tau_2^{ns}}{\partial y} \to \frac{\partial \tau_n''}{\partial y}; \; \frac{\partial \tau_1^{ns}}{\partial x} \to \frac{\partial \tau_n'}{\partial x}; \; \frac{\partial u_2^{ns}}{\partial y} \to \frac{\partial u_n''}{\partial y}; \; \frac{\partial v_2^{ns}}{\partial x} \to \frac{\partial v_n''}{\partial x}
\end{gathered} \tag{8.6}$$

which converge weakly in $L_2(\Omega)$ for fixed n as $s \to \infty$. The limit functions will satisfy the equations

$$\int_\Omega \left(\frac{\partial p_n}{\partial x}+\frac{\partial \tau_n''}{\partial y}-f_1^n\right)P_k\varphi_1 d\Omega=0,$$

$$\int_\Omega \left(\frac{\partial \tau_n'}{\partial x}+\frac{\partial q_n}{\partial y}-f_2^{n-1}\right)P_k\varphi_2 d\Omega=0,$$

$$\int_\Omega \left(p_n-\alpha\frac{\partial u_n'}{\partial x}-\beta\frac{\partial v_n'}{\partial y}+f_3^n\right)P_k\varphi_3 d\Omega=0,$$

$$\int_\Omega \left(q_n-\alpha\frac{\partial v_n'}{\partial y}-\beta\frac{\partial u_n'}{\partial x}+f_4^n\right)P_k\varphi_4 d\Omega=0, \tag{8.7}$$

$$\int_\Omega \left[\tau_n''-\mu\left(\frac{\partial u_n''}{\partial y}+\frac{\partial v_n''}{\partial x}\right)+f_5^{n+1}\right]P_k\varphi_5 d\Omega=0,\ k=0,1,\dots,$$

$$\int_\Omega \left[\tau_n'-\mu\left(\frac{\partial u_n''}{\partial y}+\frac{\partial v_n''}{\partial x}\right)+f_5\right]P_i\varphi_6 d\Omega=0,\quad i=0,1,\dots,n-1,$$

where $\varphi_r=\varphi_r(x)$ ($r=1,2,\ \dots,\ 6$) are arbitrary functions and belong to $L_2[-1,\ 1]$.

We denote by S_* the set of continuous and continuously differentiable functions p_*, q_*, τ_*, τ_*'', u_*', u_*'', v_*', v_*'', which satisfy the conditions

$$(p_* u_*')_{x=\pm 1}=(q_* v_*')_{y=\pm 1}=(\tau_*' v_*'')_{x=\pm 1}=(\tau_*'' u_*'')_{y=\pm 1}=0. \tag{8.8}$$

We assume that in each of the equations (8.8) one of the factors [the same one as in (1.1)] vanishes all along a side of the square.

We denote by S_{**} the set of functions

$$p_{**},\ q_{**},\ \tau_{**}',\ \tau_{**}'',\ u_{**}',\ u_{**}'',\ v_{**}',\ v_{**}'', \tag{8.9}$$

which satisfy the equations

$$\int_\Omega\left(\frac{\partial p_{**}}{\partial x}u_*'+p_{**}\frac{\partial u_*'}{\partial x}\right)d\Omega=0,\ \int_\Omega\left(\frac{\partial q_{**}}{\partial y}v_*'+q_{**}\frac{\partial v_*'}{\partial y}\right)d\Omega=0,$$

$$\int_\Omega\left(\frac{\partial \tau_{**}'}{\partial x}v_*''+\tau_{**}'\frac{\partial v_*''}{\partial x}\right)d\Omega=0,\ \int_\Omega\left(\frac{\partial \tau_{**}''}{\partial y}u_*''+\tau_{**}''\frac{\partial u_*''}{\partial y}\right)d\Omega=0,$$

$$\int_\Omega\left(\frac{\partial p_*}{\partial x}u_{**}'+p_*\frac{\partial u_{**}'}{\partial x}\right)d\Omega=0,\ \int_\Omega\left(\frac{\partial q_*}{\partial y}v_{**}'+q_*\frac{\partial v_{**}'}{\partial y}\right)d\Omega=0, \tag{8.10}$$

$$\int_\Omega\left(\frac{\partial \tau_*'}{\partial x}v_{**}''+\tau_*'\frac{\partial v_{**}''}{\partial x}\right)d\Omega=0,\ \int_\Omega\left(\frac{\partial \tau_*''}{\partial y}u_{**}''+\tau_*''\frac{\partial u_{**}''}{\partial y}\right)d\Omega=0,$$

where p_*, q_*, τ_*', τ_*'', u_*', u_*'', v_*', v_*'' are any functions belonging to S_*. We assume that the functions (8.9) have the quadratic summable generalized derivatives which appear in (8.10).

The approximate solution (2.1) belongs to S_*. Substituting the functions (2.1) into (8.10) and passing to the limit as $s\to\infty$, we find that the limit functions of the subsequences (8.6) satisfy the equations

$$\int_\Omega\left(\frac{\partial p_{**}}{\partial x}u_n'+p_{**}\frac{\partial u_n'}{\partial x}\right)d\Omega=0,\ \int_\Omega\left(\frac{\partial q_{**}}{\partial y}v_n'+q_{**}\frac{\partial v_n'}{\partial y}\right)d\Omega=0,$$

$$\int_\Omega\left(\frac{\partial \tau_{**}'}{\partial x}v_n''+\tau_{**}'\frac{\partial v_n''}{\partial x}\right)d\Omega=0,\ \int_\Omega\left(\frac{\partial \tau_{**}''}{\partial y}u_n''+\tau_{**}''\frac{\partial u_n''}{\partial y}\right)d\Omega=0,$$

$$\int_\Omega \left(\frac{\partial p_n}{\partial x} u'_{**} + p_n \frac{\partial u'_{**}}{\partial x}\right) d\Omega = 0, \int_\Omega \left(\frac{\partial q_n}{\partial y} v'_{**} + q_n \frac{\partial v'_{**}}{\partial y}\right) d\Omega = 0,$$
$$\int_\Omega \left(\frac{\partial \tau'_n}{\partial x} v''_{**} + \tau'_n \frac{\partial v''_{**}}{\partial x}\right) d\Omega = 0, \int_\Omega \left(\frac{\partial \tau''_n}{\partial y} u''_{**} + \tau''_n \frac{\partial u''_{**}}{\partial y}\right) d\Omega = 0, \tag{8.11}$$

where p_{**}, q_{**}, ..., v''_{**} are any functions belonging to S_{**}.

The functions (8.1) satisfying the equations (8.7), (8.11) form the generalized solution of the boundary-value problem for Eqs. (8.2), (8.4) with the boundary conditions (8.3).

Assume that the problem of Eqs. (8.2) and (8.4), with the conditions (8.3), has two generalized solutions. Let

$$p_n^0, q_n^0, \tau_n'^0, \tau_n''^0, u_n'^0, u_n''^0, v_n'^0, v_n''^0 \tag{8.12}$$

be the differences of these solutions. Since $S_* \subset S_{**}$, it follows from (8.11) that the function (8.12) belongs to S_{**}. If in (8.11) we substitute for p_{**}, q_{**}, ..., v''_{**}, p_n, q_n, ..., v''_n the corresponding functions (8.12) and make use of the fact that the functions (8.12) satisfy the equations (8.7) for $f_\sigma^r = 0$, $\sigma = 1, 2, \ldots, 6$ we find that

$$G_1(u_n'^0, v_n'^0) + G_2(u_n''^0, v_n''^0) = 0. \tag{8.13}$$

Obviously the functions (8.12) satisfy the inequalities obtained when we replace the functions (2.1) in (4.8), (4.10), (4.11), and (5.4) with the functions (8.12). From these inequalities and (8.13) it follows that the generalized solution of the problem for Eqs. (8.2) and (8.4) with the conditions (8.3) is unique. From the uniqueness of the solution it follows that the entire sequence of solutions (2.1) converges to it weakly in $L_2(\Omega)$ for fixed n as $m \to \infty$.

If the plane mixed problem of the theory of elasticity has a sufficiently smooth solution, then the entire sequence of generalized solutions of Eqs. (8.2), (8.4) with the conditions (8.3) converges to it weakly in $L_2(\Omega)$ as $n \to \infty$. The proof is analogous to the proof of the convergence of the solutions (2.1).

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